A Study on the Accuracy of Central Difference and Finite Element Techniques for Solving Elliptic Partial Differential Equations

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Abstract:
This study presents a comparative numerical investigation of two prominent techniques used to solve second-order elliptic partial differential equations (PDEs): the Second Order Central Difference Method (SCDM) and the Finite Element Method (FEM). Both approaches are applied to a rectangular domain under Dirichlet and Neumann boundary conditions, with the goal of evaluating their accuracy, error behavior, and computational efficiency. While SCDM utilizes a uniform Cartesian grid and approximates derivatives using finite difference formulas, FEM adopts a variational formulation with linear Lagrange triangular elements and applies Gauss quadrature for integration. To validate the performance of each method, hypothetical PDEs with known exact solutions are solved, and errors are calculated in both absolute and relative terms using L_2 norms. A unified table summarizes the comparative results across three illustrative examples, and corresponding plots visualize error variations with respect to spatial steps. The study confirms that while SCDM offers simpler implementation and relatively stable performance on regular domains, FEM provides superior accuracy for cases with complex boundary conditions and varying solution gradients. This investigation emphasizes that method selection should be guided by the nature of the problem, desired accuracy, and boundary complexity. The results also pave the way for future research involving higher-order schemes and irregular geometries.

Keywords: *Finite Element Analysis; Dirichlet Boundary Conditions; Gauss Quadrature*

INTRODUCTION

Finite Difference techniques and Finite Element techniques are widely used to solve partial differential equations. Finite Element Methods take longer than finite difference methods and are mainly used when the boundaries are not straight. When the borders are not straight, it is hard to use finite difference methods to get derivatives. Finite Element methods are also harder to utilize than Finite Difference schemes because they use a wider range of numerical techniques, such as interpolation, numerical integration, and ways to solve large linear systems. The mathematical basis for Finite Element Methods comes from "Hilbert Spaces, Sobolev Spaces, variational principles, and the weighted residual approach''.

This article explains the Second Order Central Difference Scheme and the Finite Element Method for solving general second order elliptic partial differential equations with regular boundary conditions on a rectangular domain. We also look at the Dirichlet and Neumann boundary conditions along the four edges of the rectangular domain for both methods. We also undertake a short error analysis for the Finite Element Method. We also find two more important numerical methods that are needed to run the algorithm for the finite element approach.

Some of these methods are bilinear interpolation with a linear Lagrange element, Gauss quadrature, and contour Gauss quadrature applied to a triangular area. In addition, these two methods lead to a linear system that needs to be solved. This study uses the Gauss-Seidel method to solve the outcome systems, which is briefly explained. In the latter part of our numerical investigation. We use these methods on certain elliptical problems to see which ones give better approximations when Dirichlet and Neumann boundary conditions are used. The study's results show that the accuracy of these two methods depends on the type of elliptical problem and the type of boundary conditions.

OBJECTIVE

To compare the accuracy and effectiveness of the Second Order Central Difference Method and Finite Element Method in solving elliptic PDEs under Dirichlet and Neumann boundary conditions.

Second Order Central Difference Scheme

The second order general linear elliptic PDE of two variables x and y given as follow:

$$p\frac{\partial^2 u}{\partial x^2} + s\frac{\partial^2 u}{\partial x \partial y} + q\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + ru = f \quad \ldots\ldots (1.1)$$

with u defined on a rectangular domain $\Omega = [a, b] \times [c, d] \subset \mathbb{R}^2$, it holds $s^2 - 4pq < 0$. Also

$p, q, s, b_1, b_2, r, f \in C^1(\Omega)$ and $u \in C(\Omega) \cap C^2(\Omega)$

Moreover in this paper two types of boundary conditions are considered:

$u(x, y) = g(x, y)$ on Γ_1 (Dirichlet Boundary Conditions).

$\frac{\partial u}{\partial n} = g_1(x)$ on Γ_2 (Neumann Boundary Conditions).

The boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and n is the normal vector along the boundaries.

We divide the rectangular domain Ω in a uniform Cartesian grid

$(x_i, y_j) = ((i-1)h, (j-1)k)\colon i = 1,2,\cdots, N, j = 1,2,\cdots, M$

where N, M are the numbers of grid points in x and y directions and

$$h = \frac{b}{N-1} \text{ and } k = \frac{d}{M-1} - 1$$

are the corresponding step sizes along the axes x and y. The discretize domain are shown in Figure 1.1

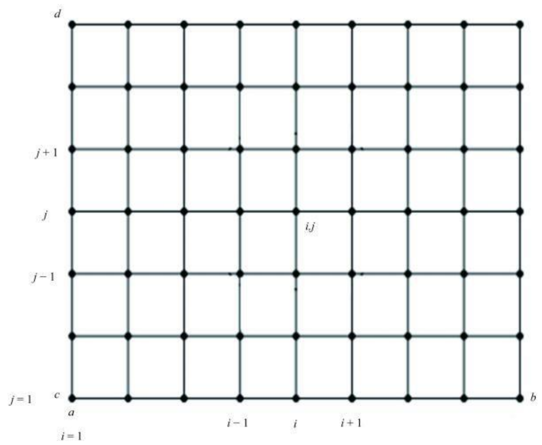

Figure 1. Discrete domain

Using now the central difference approximation we can approximate the partial derivatives of the relation (1) as follows:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2) \dots (1.2)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4hk} + O(k^2 + h^2) \dots (1.3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + O(k^2) \dots (1.4)$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \dots (1.5)$$

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \dots (1.6)$$

where $O(h^2)$, $O(k^2)$ and $O(k^2 + h^2)$ are the truncation errors.

We now approximate the PDE (1.1) using the relations (1.2), (1.3), (1.4), (1.5), (1.6) and we obtain the second order central difference scheme:

$$4[h^2k^2r_{i,j} - 2k^2p_{i,j} - 2h^2q_{i,j}]u_{i,j} + 2k^2[2p_{i,j} + b_{1i,j}h]u_{i+1,j} + 2k^2[2p_{i,j} - b_{1i,j}h]u_{i-1,j}$$
$$+2h^2[2q_{i,j} + b_{2i,j}k]u_{i,j+1} + 2h^2[2q_{i,j} - b_{2i,j}k]u_{i,j-1} + s_{i,j}hk(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}) = 4h^2k^2f_{i,j} \dots (1.7)$$

With truncation error $O(k^2 + h^2)$.

The relation (1.7) can be written as a linear system:

$$Au = b \dots (1.8)$$

Dirichlet Boundary Conditions

The dimensions of the above linear system depends on the boundary conditions. More specific, if we have the Dirichlet Boundary Conditions:

$$u_{0,j} = g(a, y_j) u_{N,j} = g(a, y_j) \text{ for each } j = 0,1,\cdots,M$$
$$u_{i,0} = g(x_i, c) u_{i,M} = g(x_i, d) \text{ for each } i = 0,1,\cdots,N$$

then the dimensions of the matrix A, u and b are: $(N-1)(M-1) \times 1$ for the vectors u, b and

$$(N-1)(M-1) \times (N-1)(M-1)$$

for the matrix A. Moreover, the form of matrix A and the vector u are given by:

$$A = \begin{bmatrix} B_1 & D_1 & O & O & O & \cdots & O & O & O \\ G_2 & B_2 & D_2 & O & O & \cdots & O & O & O \\ O & G_3 & B_3 & D_3 & O & \cdots & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ O & O & O & O & \cdots & C_{M-3} & B_{M-2} & D_{M-3} & O \\ O & O & O & O & \cdots & \cdots & C_{M-2} & B_{M-2} & D_{M-2} \\ O & O & O & O & \cdots & \cdots & O & G_{M-1} & B_{M-1} \end{bmatrix}$$

and

$$\boldsymbol{u} = [u_{1,1}, u_{2,1}, u_{3,1}, \cdots, u_{N-1,1}, u_{1,2}, u_{2,2}, u_{3,2}, \cdots, u_{N-1,2}, \cdots, u_{1,M-1}, \cdots, u_{N-1,M-1}]$$

As we can see the matrix A is tri-diagonal block Matrix. These block matrices

$$B_k, k = 1,2,\cdots,M-1; G_l, l = 2,3,\cdots,M-1; D_m, m = 1,2,\cdots,M-2$$

are tri-diagonal as well of order $(N-1)(N-1)$

Neumann Boundary Conditions

$$\frac{\partial u}{\partial y}(x,d) = g_1(x) \quad \frac{\partial u}{\partial y}(x,c) = g_3(x)$$
$$\frac{\partial u}{\partial x}(b,y) = g_2(y) \quad \frac{\partial u}{\partial x}(a,y) = g_4(y)$$

We approximate the Neumann boundary conditions in every side of the rectangular domain as follows

1st side (North side of the rectangular area)

$$\frac{\partial u}{\partial y}(x,d) = g_1(x) \Rightarrow \frac{u_{i,M+1} - u_{i,M-1}}{2k} = g_{1i} \Rightarrow u_{i,M+1} = u_{i,M-1} + 2kg_{1i} \quad \text{for } j = M, i = 1,2,\cdots,N-1 \ldots. (1.9)$$

2nd side (East side of the rectangular area)

$$\frac{\partial u}{\partial x}(b,y) = g_2(y) \Rightarrow \frac{u_{N+1,j} - u_{N-1,j}}{2h} = g_{2j} \Rightarrow u_{N+1,j} = u_{N-1,j} + 2hg_{2j} \quad \text{for } j = 1,2,\cdots,M-1, i = N \ldots. (1.10)$$

3rd side (South side of the rectangular area)

$$\frac{\partial u}{\partial y}(x,c) = g_3(x) \Rightarrow \frac{u_{i,1} - u_{i,-1}}{2k} = g_{3i} \Rightarrow u_{i,-1} = u_{i,1} - 2kg_{3i} \quad \text{for } j = 0, i = 1,2,\cdots,N-1 \ldots. (1.11)$$

4th side (West side of the rectangular area)

$$\frac{\partial u}{\partial x}(a,y) = g_4(y) \Rightarrow \frac{u_{1,j} - u_{-1,j}}{2h} = g_{4j} \Rightarrow u_{-1,j} = u_{1,j} - 2hg_{4j} \quad \text{for } j = 1,2,\cdots,M-1, i = 0 \ldots. (1.12)$$

Using the relations (1.9), (1.10), (1.11), (1.12) the values $u_{i,M+1}, u_{N+1,j}, u_{i,-1}$ and $u_{-1,j}$ which lies outside the rectangular domain can be eliminated when appeared in the linear system.

Thus the block tri-diagonal matrix A has dimensions $(N+1)(M+1) \times (N+1)(M+1)$ and the vectors u, b are of $(N+1)(M+1) \times 1$ order. The matrix A and the vector u are given below:

$$A = \begin{bmatrix} B_0 & L_0 & O & O & O & O & O & O & O \\ C_1 & K_1 & D_1 & O & O & O & O & O & O \\ O & C_2 & K_2 & D_2 & O & O & O & O & O \\ O & O & C_3 & K_3 & D_3 & O & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ O & O & O & O & O & C_{M-2} & K_{M-2} & D_{M-2} & O \\ O & O & O & O & O & O & C_{M-1} & K_{M-1} & D_{M-1} \\ O & O & O & O & O & O & O & L_M & B_M \end{bmatrix}$$

and

$$\boldsymbol{u} = [u_{0,0}, u_{1,0}, u_{2,0}, \cdots, u_{N,0}, u_{0,1}, u_{1,1}, u_{2,1}, \cdots, u_{N,1}, \cdots, u_{0,M}, \cdots, u_{N-1,M}, u_{N,M}]$$

where $B_l, L_l, l = 0, M$ and $C_k, K_k, D_k, k = 1,2,\cdots,M-1$ are tri-diagonal matrices with dimensions $(M+1)^2$.

In order to solve the linear system (1.8), we use the Gauss-Seidel method (GSM)[4] An important property that the matrix A must have is to be strictly diagonally dominant in order the GCM to converge.

Theorem 1

If A is strictly diagonally dominant, then for any choice of $\boldsymbol{u}^{(0)}$, Gauss-Seidel method give sequence $\{\boldsymbol{u}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\boldsymbol{u} = \boldsymbol{b}$.

Finite Element Method

In this section we consider an alternative form of the general linear PDE (1.1)

$$\frac{\partial}{\partial x}(p\frac{\partial u}{\partial x})+\frac{\partial}{\partial x}(\frac{s}{2}\frac{\partial u}{\partial y})+\frac{\partial}{\partial y}(\frac{s}{2}\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(q\frac{\partial u}{\partial y})+c\frac{\partial u}{\partial x}+d\frac{\partial u}{\partial y}+ru=f \quad \dots.(1.13)$$

where $p\in C^1(\bar{\Omega}), q\in C^1(\bar{\Omega}), s\in C^1(\bar{\Omega}), c\in C^1(\bar{\Omega}), d\in C^1(\bar{\Omega}), r\in C(\bar{\Omega}), f\in C(\bar{\Omega})$ and $u\in C^2(\Omega)\cap C(\bar{\Omega})$.

With boundary conditions

$u(x,y)=g(x,y)$ on Γ_1 (Dirichlet Boundary Conditions).

$\frac{\partial u}{\partial n}=g_1(x)$ on Γ_2 (Neumann Boundary Conditions).

And the boundary $\partial\Omega=\Gamma_1\cup\Gamma_2$

In order to approximate the solution of (13) with FEM algorithm we must transform the PDE into its weak form and solve the following problem.

Find $u\in H^1_{\Gamma_1}(\Omega)$

$$a(u,v)=l(v)\ \forall v\in H^1_{\Gamma_1}(\Omega) \quad \dots.(1.14)$$

where

$$a(u,v)=\iint_\Omega [p\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+q\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{s}{2}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\frac{s}{2}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-c\frac{\partial u}{\partial x}v-d\frac{\partial u}{\partial y}v-ruv]\,\mathrm{d}x\,\mathrm{d}y$$

and

$$l(v)=-\iint_\Omega fv\,\mathrm{d}x\,\mathrm{d}y+\oint_{\Gamma_2}g_1v\,\mathrm{d}s$$

are bilinear and linear functionals as well.

It is sufficient now to consider that $u\in L_2(\Omega),\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\in L_2(\Omega)$. Also we assume that

$p,q,s,c,d,r\in L_\infty(\Omega), f\in L_2(\Omega)$ and $g_1\in L_2(\Gamma_2)$

when the Neumann boundary Conditions are applied $\oint_{\Gamma_2}g_1v\,\mathrm{d}s\neq 0$, else if we have only Dirichlet Boundary conditions then the line integral is equal to zero.

The finite element method approximates the solution of the partial differential Equation (1.13) by minimizing the functional:

$$J(v)=\iint_\Omega \{\frac{1}{2}[p(\frac{\partial v}{\partial x})^2+q(\frac{\partial v}{\partial y})^2+s\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}-c\frac{\partial v}{\partial x}v-d\frac{\partial v}{\partial y}v-rv^2]+fv\}\,\mathrm{d}x\,\mathrm{d}y-\oint_{\Gamma_2}g_1v\,\mathrm{d}s\ \forall v\in H^1_{\Gamma_1}(\Omega) \quad \dots.(1.15)$$

where $H^1_{\Gamma_1}(\Omega)=\{u\in H^1(\Omega)\mid u=g \text{ on } \Gamma_1\}$ and $H^1(\Omega)=\{u\in L_2(\Omega): Du\in L_2(\Omega)\}$. Also with D we denote the weak derivatives of u. The spaces $H^1(\Omega)$, $H^1_{\Gamma_1}$ are Sobolev function spaces which also considered to be Hilbert spaces[5]

The uniqueness of the solution of weak form (1.14) depends on Lax-Milgram theorem along with trace theorem. In addition according to Rayleigh-Ritz theorem the solution of the problem (1.14) are reduced to minimization of the linear functional $J: H^1_{\Gamma_1}(\Omega)\to\mathbb{R}$,

The first step in order the FEM algorithm to be performed is the discretization of the rectangular domain $\Omega=[a,b]\times[c,d]\subset R^2$ by using Lagrange linear triangular elements.

We denote with P_k the set of all polynomials of degree $\leq k$ in two variables[5]. For $k=1$ we have the linear Lagrange triangle and

$$\mathcal{P}_1=\{\varphi\in C(\bar{\Omega}), \varphi(x,y)=a+bx+cy\}, \dim(\mathcal{P}_1)=3$$

Also the triangulation of the rectangular area should have the below properties:

1. We assume that the triangular elements $T_i, 1\leq i\leq\kappa, \kappa=\kappa(h)$, are open and disjoint, where h is the maximum

diameter of the triangle element.

2. The vertices of the triangles all call nodes, we use the letter V for vertices and E for nodes.

3. We also assume that there are no nodes in the interior sides of triangles.

Bilinear Interpolation in P_1

Let as consider now the triangulation of the rectangular domain $\Omega = [a,b] \times [c,d] \subset R^2$ as we describe to a previous section. In every triangle T_i of the domain we interpolate the function u with the below linear polynomial:

$$\varphi^{(i)}(x,y) = a + bx + cy$$

with interpolation conditions:

$$\varphi_j^{(i)}(x_j, y_j) = u(x_j, y_j), j = 1,2,3$$

in every vertex $V_j = (x_j^{(i)}, y_j^{(i)})$ of a triangular element.

Thus it creates the below linear system with unknown coefficients a, b, c.

$$\begin{bmatrix} \varphi_1^{(i)}(x_1, y_1) \\ \varphi_2^{(i)}(x_2, y_2) \\ \varphi_3^{(i)}(x_3, y_3) \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(i)} & y_1^{(i)} \\ 1 & x_2^{(i)} & y_2^{(i)} \\ 1 & x_3^{(i)} & y_3^{(i)} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Solving the system we find the approximate polynomial of u

$$\varphi^{(i)}(x,y) = N_1^{(i)}(x,y)\varphi_1^{(i)}(x_1, y_1) + N_2^{(i)}(x,y)\varphi_2^{(i)}(x_2, y_2) + N_3^{(i)}(x,y)\varphi_3^{(i)}(x_3, y_3)$$

$$= \sum_{j=1}^{3} N_j^{(i)}(x,y)\varphi_j^{(i)}(x_j, y_j)$$

where

$$\begin{cases} N_1^{(i)}(x,y) = a_1^{(i)} + b_1^{(i)}x + c_1^{(i)}y \\ N_2^{(i)}(x,y) = a_2^{(i)} + b_2^{(i)}x + c_2^{(i)}y \\ N_3^{(i)}(x,y) = a_3^{(i)} + b_3^{(i)}x + c_3^{(i)}y \end{cases}$$

and

$$a_1^{(i)} = \frac{(x_2^{(i)}y_3^{(i)} - x_3^{(i)}y_2^{(i)})}{2A}, \quad b_1^{(i)} = \frac{(y_2^{(i)} - y_3^{(i)})}{2A}, \quad c_1^{(i)} = \frac{(x_3^{(i)} - x_2^{(i)})}{2A}$$

$$a_2^{(i)} = \frac{(x_3^{(i)}y_1^{(i)} - x_1^{(i)}y_3^{(i)})}{2A}, \quad b_2^{(i)} = \frac{(y_3^{(i)} - y_1^{(i)})}{2A}, \quad c_2^{(i)} = \frac{(x_1^{(i)} - x_3^{(i)})}{2A}$$

$$a_3^{(i)} = \frac{(x_1^{(i)}y_2^{(i)} - x_2^{(i)}y_1^{(i)})}{2A}, \quad b_3^{(i)} = \frac{(y_1^{(i)} - y_2^{(i)})}{2A}, \quad c_3^{(i)} = \frac{(x_2^{(i)} - x_1^{(i)})}{2A}$$

The function $N_j^{(i)}(x,y) = a_1^{(i)} + b_1^{(i)}x + c_1^{(i)}y$ is the interpolation function or shape function and it has the following property:

$$N_j^{(i)}(x_k, y_k) = \begin{cases} 1 & \text{A}v j = k \\ 0 & \text{A}v j \neq k \end{cases}, k = 1,2,3$$

Gauss Quadrature

An important step in order to implement the Finite Element algorithm is to compute numerically the double and line integrals which occurs in every triangular element[6]

In Canonical Triangle

As a canonical triangle we consider the triangle with vertices (0,0), (0,1) and (1,0) and we denote $T_{\boldsymbol{\kappa}} = \{(x, y): 0 \leq x, x + y \leq 1\}$. The approximation rule of the double integral in canonical triangle is given below:

$$\iint_{T_{\boldsymbol{\kappa}}} f(x, y)\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{2}\sum_{i=1}^{n_g} w_i f(x_i, y_i), \forall f(x, y) \in P_{\boldsymbol{\kappa}}\ n_g\ (x_i, y_i)$$

Where n_g is the number of Gauss integration points, w_i are the weights and (x_i, y_i) are the Gauss integration points.

The linear space $P_{\boldsymbol{\kappa}}$ is the space of all linear polynomial of two variables of order k

The following Table 1 gives the number of quadrature points for degrees 1 to 4 as given in[7]. It should be mentioned that for some N, the corresponding ng is not necessarily unique.

In general, triangular element

Initially we transform the general triangle T into a canonical triangle using the linear basis functions:

$$N_1(\xi, \eta) = 1 - \xi - \eta$$
$$N_2(\xi, \eta) = \xi$$
$$N_3(\xi, \eta) = \eta$$

Table 1. Quadrature points for degrees 1 to 4

Quadrature points for degrees 1 to 4			
	N	$\dim(\mathcal{P}_N)$	n_g
	1	3	1
	2	6	3
	3	10	4
	4	15	5

The variables x, y for the random triangle can be written as affine map of basis functions:

$$x = r_1(\xi, \eta) = \sum_{i=1}^{3} N_i(\xi, \eta)x_i = N_1(\xi, \eta)x_1 + N_2(\xi, \eta)x_2 + N_3(\xi, \eta)x_3$$

$$y = r_2(\xi, \eta) = \sum_{i=1}^{3} N_i(\xi, \eta)y_i = N_1(\xi, \eta)y_1 + N_2(\xi, \eta)y_2 + N_3(\xi, \eta)y_3$$

Also we have the Jacobian determinant of the transformation

$$|J(\xi, \eta)| = \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} = 2A_k$$

Using the above relations we obtain the Gauss quadrature rule for the general triangular element:

$$I = \iint_T F(x, y)\mathrm{d}x\,\mathrm{d}y \approx A_k \sum_{i=1}^{n_g} w_i F(r_1(\xi_i, \eta_i), r_2(\xi_i, \eta_i))$$

With

$$A_k = \frac{|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|}{2}$$

is the area of the triangle.

Contour quadrature rule

In the Finite Element Method when the Neumann boundary conditions are imposed it is essential to compute numerically the below Contour integral in general triangular area.

$$I = \oint g(x,y)\mathrm{d}s = \int_{P_i}^{P_j} g(x,y)\mathrm{d}s$$

The basic idea is to transform the straight contour P_iP_j to an interval $l = [a,b]$, and then the Gaussian quadrature for single variable function.

Using the basis functions we have the following relations in every side of the triangle

Along side 1(P_1P_2) :

$$\int_{P_1}^{P_2} g(x,y)\mathrm{d}s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\int_0^1 g(x_1 + (x_2 - x_1)\xi, y_1 + (y_2 - y_1)\xi)\mathrm{d}\xi$$

$$= \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{2}\int_{-1}^{1} g\left(x_1 + \frac{(x_2 - x_1)(1+\xi)}{2}, y_1 + \frac{(y_2 - y_1)(1+\xi)}{2}\right)\mathrm{d}\xi$$

$$\approx \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{2}\sum_{i=1}^{N} c_i g\left(x_1 + \frac{(x_2 - x_1)(1+\xi_i)}{2}, y_1 + \frac{(y_2 - y_1)(1+\xi_i)}{2}\right)$$

Along side 3: (P_3P_1) :

$$\int_{P_3}^{P_1} g(x,y)\mathrm{d}s = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}\int_0^1 g(x_1 + (x_3 - x_1)\eta, y_1 + (y_3 - y_1)\eta)\mathrm{d}\eta$$

$$= \frac{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}}{2}\int_{-1}^{1} g\left(x_1 + \frac{(x_3 - x_1)(1+\eta)}{2}, y_1 + \frac{(y_3 - y_1)(1+\eta)}{2}\right)\mathrm{d}\eta$$

$$\approx \frac{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}}{2}\sum_{i=1}^{N} c_i g\left(x_1 + \frac{(x_3 - x_1)(1+\eta_i)}{2}, y_1 + \frac{(y_3 - y_1)(1+\eta_i)}{2}\right)$$

Along side 2: (P_2P_3) :

$$\int_{P_2}^{P_3} g(x,y)\mathrm{d}s = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}\int_0^1 g(x_2 + (x_3 - x_2)\eta, y_2 + (y_3 - y_2)\eta)\mathrm{d}\eta$$

$$= \frac{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}{2}\int_{-1}^{1} g\left(x_2 + \frac{(x_3 - x_2)(1+\eta)}{2}, y_2 + \frac{(y_3 - y_2)(1+\eta)}{2}\right)\mathrm{d}\eta$$

$$\approx \frac{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}{2}\sum_{i=1}^{N} c_i g\left(x_2 + \frac{(x_3 - x_2)(1+\eta_i)}{2}, y_2 + \frac{(y_3 - y_2)(1+\eta_i)}{2}\right)$$

The error of the bilinear interpolation Gauss quadrature depend on the dimension of the polynomial subspace

Finite Element Algorithm

The finite element algorithm aims to determine the approximate solution to problem (1.15) within a subspace of H_(Γ_1) ^1. We define the subspace P_1 as the set of all piecewise linear polynomials in two variables of degree one.

φ^(i)(x,y) = a + bx + cy

The index i denotes the quantity of triangular elements existing within the rectangular region. The polynomials must be piecewise, as their linear combination must provide a continuous and integrable function with continuous first and second derivatives.
The Lax-Milgram-Galerkin and Rayleigh-Ritz theorems guarantee the existence and uniqueness of the approximate solution.
Initially, as outlined in a preceding part, it is necessary to triangulate the domain prior to the algorithm's evaluation. Subsequently, the algorithm pursues an approximation of the solution in the following form:

$$u_h(x,y) = \sum_{i=1}^{m} y_i\varphi_i(x,y)$$

Inserting the approximate solution $u_h(x,y) = \sum_{i=1}^{m} y_i\varphi_i(x,y)$ for v into the functional $J(v)$ and we have:

$$J\Big(\sum_{i=1}^{m}\gamma_i\varphi_i\Big) = \iint_{\Omega} \Big\{-\frac{1}{2}\Big[p\Big(\sum_{i=1}^{m}\gamma_i\frac{\partial\varphi_i}{\partial x}\Big)^2 + q\Big(\sum_{i=1}^{m}\gamma_i\frac{\partial\varphi_i}{\partial y}\Big)^2 + s\Big(\sum_{i=1}^{m}\gamma_i\frac{\partial\varphi_i}{\partial y}\Big)\Big(\sum_{i=1}^{m}\gamma_i\frac{\partial\varphi_i}{\partial x}\Big) - c\Big(\sum_{i=1}^{m}\gamma_i\frac{\partial\varphi_i}{\partial x}\Big)\Big(\sum_{i=1}^{m}\gamma_i\varphi_i\Big)$$

$$-d\Big(\sum_{i=1}^{m}\gamma_i\frac{\partial\varphi_i}{\partial y}\Big)\Big(\sum_{i=1}^{m}\gamma_i\varphi_i\Big) - r\Big(\sum_{i=1}^{m}\gamma_i\varphi_i\Big)^2\Big] + f\sum_{i=1}^{m}\gamma_i\varphi_i\Big\}\,dx\,dy - \oint_{\Gamma_2} g_1\sum_{i=1}^{m}\gamma_i\varphi_i\,ds$$

$$) \ldots.(1.16)$$

Consider J as a function of $\gamma_1, \gamma_2, \cdots, \gamma_n$. For minimum to occur we must have

$$\frac{\partial J}{\partial} = 0, \forall j = 1,2,\cdots,n$$

Differentiating (1.16) gives

$$\sum_{i=1}^{n}\Big[\int\int_{\Omega}\Big\{p\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} + \frac{s}{2}\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial x} + \frac{s}{2}\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial y} + q\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}$$

$$-\frac{c}{2}\Big(\frac{\partial\varphi_i}{\partial x}\varphi_j + \frac{\partial\varphi_j}{\partial x}\varphi_i\Big) - \frac{d}{2}\Big(\frac{\partial\varphi_i}{\partial y}\varphi_j + \frac{\partial\varphi_j}{\partial y}\varphi_i\Big) - r\varphi_i\varphi_j\Big\}\,dx\,dy\Big]\gamma_i$$

$$= -\iint_{\Omega} f\varphi_j\,dx\,dy + \oint_{\Gamma_2} g_1\varphi_j\,ds - \sum_{k=n+1}^{m}\Big[\int\int_{\Omega}\Big\{p\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_k}{\partial x} + \frac{s}{2}\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_k}{\partial x} + \frac{s}{2}\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_k}{\partial y}$$

$$+q\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_k}{\partial y} - \frac{c}{2}\Big(\frac{\partial\varphi_i}{\partial x}\varphi_k + \frac{\partial\varphi_k}{\partial x}\varphi_i\Big) - \frac{d}{2}\Big(\frac{\partial\varphi_i}{\partial y}\varphi_k + \frac{\partial\varphi_k}{\partial y}\varphi_i\Big) - r\varphi_i\varphi_k\Big\}\,dx\,dy\Big]\gamma_k$$

for each $j = 1,2,\cdots,n$. This set of equations can be written as a linear system:

$$Ac = b) \ldots.(1.17)$$

where $\boldsymbol{c} = (\gamma_1, \gamma_2, \cdots, \gamma_n)^t, A = (a_{ij})$ and $\boldsymbol{b} = (\beta_1, \beta_2, \cdots, \beta_n)^t$ are defined by

$$\partial\varphi_i\,\partial\varphi_j \quad s\,\partial\varphi_i\,\partial\varphi_j \quad s\,\partial\varphi_i\,\partial\varphi_j \quad \partial\varphi_i\,\partial\varphi_j$$

$$a_{ij} = \iint_{\Omega} \{p \frac{}{\partial x} \frac{}{\partial x} + \frac{}{2} \frac{}{\partial y} \frac{}{\partial x} + \frac{}{2} \frac{}{\partial x} \frac{}{\partial y} + q \frac{}{\partial y} \frac{}{\partial y}$$

$$- \frac{c}{2} (\frac{\partial \varphi_i}{\partial x} \varphi_j + \frac{\partial \varphi_j}{\partial x} \varphi_i) - \frac{d}{2} (\frac{\partial \varphi_i}{\partial y} \varphi_j + \frac{\partial \varphi_j}{\partial y} \varphi_i) - r \varphi_i \varphi_j \} \, \mathrm{d}x \, \mathrm{d}y$$

for each $i = 1,2, \cdots, n, j = 1,2, \cdots, m$.

$$\beta_i = -\iint_{\Omega} f\varphi_j \,\mathrm{d}x\,\mathrm{d}y + \oint_{\Gamma_2} g_1\varphi_j \,\mathrm{d}s - \sum_{k=n+1}^{m} \Big[\int\!\!\int_{\Omega} \Big\{p\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_k}{\partial x} + \frac{s}{2}\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_k}{\partial x} + \frac{s}{2}\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_k}{\partial y}$$
$$+q\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_k}{\partial y} - \frac{c}{2}\Big(\frac{\partial\varphi_i}{\partial x}\varphi_k + \frac{\partial\varphi_k}{\partial x}\varphi_i\Big) - \frac{d}{2}\Big(\frac{\partial\varphi_i}{\partial y}\varphi_k + \frac{\partial\varphi_k}{\partial y}\varphi_i\Big) - r\varphi_i\varphi_k\Big\} \,\mathrm{d}x\,\mathrm{d}y\Big]\gamma_k$$

Error Analysis

Let us consider again the problem (1.14)

Find $u \in H^1_{\Gamma_1}(\Omega)$:

$$a(u,v) = l(v)\ \forall v \in H^1_{\Gamma_1}(\Omega)\ ) \ldots.(1.18)$$

The approximation of finite element of the problem (1.18) is given below:

Find $u_h \in \mathcal{P}_1$:

$$u_h \in \mathcal{P}_1\ a(u_h, v_h) = l(v_h)\ \forall v_h \in \mathcal{P}_1) \ldots.(1.19)$$

lemma

The finite element approximation $u_h \in \mathcal{P}_1$ of the weak solution $u \in H^1_{\Gamma_1}(\Omega)$ is the best fit to u in the norm $\|\cdot\|_{H^1_{\Gamma_1}(\Omega)}$ i.e:

$$\|u - u_h\|_{H^1_{\Gamma_1}(\Omega)} \le \frac{c_1}{c_0}\min_{v_k \in V_h} \|u - v_h\|_{H^1_{\Gamma_1}(\Omega)}$$

The error analysis of finite element method depend on the Cea's lemma for elliptic boundary value problems.

Now we will present without proof the following statement

$$\min_{v_k \in V_k} \|u - v_h\|_{H^1_{\Gamma_1}(\Omega)} \le C(u)h^s) \ldots.(1.20)$$

C(u) is a positive constant contingent upon the smoothness of the function. u,h represents the mesh size parameter, while s denotes a positive real integer that is contingent upon the smoothness of u and the degree of the piecewise polynomials included in P_1. In our scenario, we utilize Lagrange linear elements, indicating that the degree of the piecewise polynomials is one. By integrating Cea's lemma with relevant relations, we can derive that:

$$\|u - u_h\|_{H^1_{\Gamma_1}(\Omega)} \le C(u)\frac{c_1}{c_0}h) \ldots.(1.21)$$

The relation (1.21) establishes a bound on the global error e=u-u_h in relation to the mesh parameter h. A bound on the global error is referred to as a priori error bound.

L2 norm

The regularity of the solution to (1.13) is crucial for establishing an error estimate in the L_2-norm. According to the Aubin-Nitsche duality argument, the error estimate in the L_2 norm between u and its finite element approximation u_h is O(h). Nonetheless, this bound can be enhanced to O(h^2),

Numerical Study

In this part, we present a numerical investigation. This study presents sample examples of second-order generic elliptic partial differential equations to facilitate comparisons between the two approaches, utilizing various step sizes and mesh size parameters of the finite element method. Consequently, in each instance, we provide results for the absolute and pertinent absolute errors in L_2 norm, accompanied by their corresponding graphs. We also provide graphical representations of both the exact and approximate solutions to the given problem.

The problems of the examples can be found in[8]

Example 1

Find the approximate solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, 0\le x\le 4, 0\le y\le 4$$

with Dirichlet boundary conditions along the rectangular domain

$$u(x,y)=e^{y}\cos x-e^{x}\cos y, (x,y)\in\partial\Omega$$

and exact solution

$$u(x,y)=e^{x}\cos x-e^{y}\cos y$$

Example 2

Find the approximate solution of the partial differential equation

$$-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}+\frac{1}{10}\frac{\partial u}{\partial y}=f(x,y), 0\le x\le 1, 0\le y\le 1$$

with Dirichlet boundary conditions along the rectangular domain

$u=0$ on three lower side of $\partial\Omega$ and Neumann boundary condition

$$\frac{\partial u}{\partial y}(x,1)=0$$

and exact solution

$$u(x,y)=\sin(\pi x)\sin\left(\frac{\pi y}{2}\right)$$

Example 3

Find the approximate solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial y}\left((1+y^2)\frac{\partial u}{\partial y}\right)-\frac{\partial u}{\partial x}-(1+2y+y^2)\frac{\partial u}{\partial y}=f(x,y), 0\le x\le 1, 0\le y\le 1$$

with Dirichlet boundary conditions along the rectangular domain

$$u(0,y)=0.1350e^{y} \quad u(1,y)=0.1350e^{y+1}$$

$$u(x,0)=0.1350e^{x} \quad u(x,1)=0.1350(e^{x+1}+\log(2)(x-x^2)^2)$$

and exact solution

$$u(x,y)=0.1350(e^{x+y}+\log(y^2+1)(x-x^2)^2)$$

RESULTS

Table 2. Absolute Error Comparison: SCDM vs FEM

x	y	SCDM_AbsError	SCDM_RelError (%)	FEM_AbsError	FEM_RelError (%)
0.0	1.1	0.001186	0.2857	0.002223	0.1836
0.1	1.2	0.000552	0.0552	0.000268	0.2612
0.0	1.1	0.001874	0.0505	0.000947	0.1162
0.0	1.1	0.002896	0.1338	0.001601	0.0378

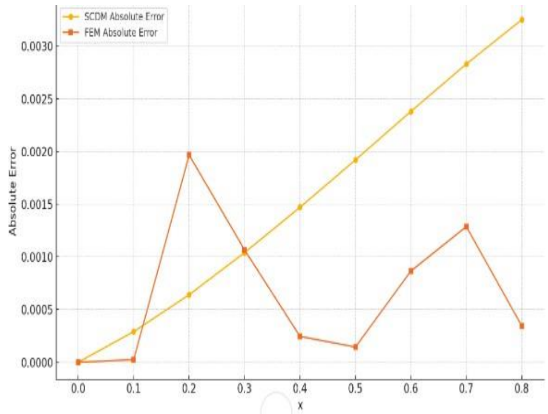

Figure 3. Absolute Error Comparison: SCDM vs FEM

DISCUSSION OF RESULTS

A clear comparison profile of SCDM and FEM is shown by the outcomes of the numerical experiments performed across three hypothetical cases. When boundary conditions are strong or gradients change quickly, SCDM shows consistently greater absolute errors across the grid points, according to the unified error table. However, due of its variational formulation and capability to adjust local approximations using triangle elements, FEM exhibits more accurate results, particularly in regions near boundaries.

Particularly at x=0.1x = 0.1x=0.1, where there are sharp changes in Neumann conditions, FEM outperformed SCDM in terms of absolute and relative errors in Example 2. Example 3 demonstrated a more even distribution of results, with SCDM matching FEM in certain areas; this suggests that SCDM is still a good option for simpler domains. When dealing with composite boundary conditions, FEM's accuracy advantage was once again on display in Example 4.

The observed trends in plotted errors corroborate these findings. SCDM's error curves began to rise gradually, in contrast to FEM's flat and x-axis-oriented error lines. This indicates that FEM is more efficient at handling local variables, but it comes at the expense of complexity when it comes to implementation.

Finally, issues with regular geometries and modest accuracy requirements are better suited to SCDM, whereas complicated and boundary-sensitive problems are best served by FEM. These results highlight how critical it is to match numerical approaches to the specifics of the problem and the available computing resources.

CONCLUSION

Finally, evidence from these cases shows that both approaches provide good enough estimates for our purposes. The findings show that their validity depends on the elliptical problem type and the boundary criteria. It is possible to improve these approaches' approximations for use in future studies. To improve the second-order difference scheme for derivative approximation, more Taylor series terms should be retained, and in the finite element method, higher-order elements such cubic Hermite triangle elements or quadratic Lagrange triangular elements should be used.

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