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A Comparative Analysis of Numerical Solutions to an Ode's Ivp Using the Euler Method and the Runge Kutta Method

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Abstract: This paper primarily introduces the Euler technique and the fourth-order Runge Kutta technique (RK4) as algorithms for solving Initial Value Problems (IVP) in the context of ordinary differential equations (ODE). The two proposed solutions exhibit a high level of efficiency and practical suitability in addressing these difficulties. To ascertain the accuracy, a comparison is made between numerical solutions and exact solutions. The obtained numerical solutions exhibit a high level of concordance with the exact solutions. Comparative analysis has been conducted to evaluate the numerical performance of the Euler method and the Runge-Kutta method. For the purpose of attaining enhanced precision in the solution, it is important to minimise the step size. In conclusion, we conduct an investigation and calculate the mistakes associated with the two proposed approaches for a specific single step size in order to assess their relative superiority.
Keywords: *IVP; ODE; Euler method; Runge Kutta method.*

INTRODUCTION

In the fields of science and engineering, differential equations are frequently employed for mathematical modelling purposes [1]. Differential equations are a common starting point for many mathematical physics problems. When other mathematical problems, such ordinary differential equations or partial differential equations, are reformulated, these equations also appear. Typically, there are two ways to get close to the solution when the underlying differential equation is too complex to solve precisely [2]. First, we can get an exact solution to the differential equation by first making it as simple as possible. Then, we can use this solution to get close to the original answer. In contrast, this study will focus on a strategy that uses approaches to approximate the solution of the original problem. Since approximation methods provide more precise findings and realistic error information, this is the most often used methodology [3]. When accurate answers to mathematical problems posed in the scientific and engineering fields are difficult, if not impossible, to obtain, numerical approaches are typically employed. Analytical solutions are available for a finite set of differential equations. Ordinary differential equations have numerous analytical solutions. There are still a lot of ordinary differential equations for which closed-form solutions are not possible using standard analytical techniques; in these cases, we need to resort to numerical methods in order to obtain an approximation of a differential equation's solution given a set of initial conditions [4]. Ordinary differential equation initial value problems can be solved using a wide variety of realistic numerical approaches. Two well-known numerical approaches for solving initial value issues of ordinary differential equations are presented in this paper: Euler and Runge Kutta.

LITERATURE REVIEW

There have been several attempts by writers to quickly and accurately answer initial value problems (IVP) using a variety of methods, including the Euler method, the Runge Kutta method, and others. Accuracy analysis of numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE) was covered by Kamruzzaman and Nath (2018) [5], and by Fadugba et al. (2020) [6], accurate solutions of IVP for ODE were discussed using the fourth-order Runge kutta method. In order to find solutions to initial value problems in ordinary differential equations Ogunrinde Fadugba (2020) [7] investigated various numerical methods. Numerical solutions of initial value problems for ordinary differential equations utilising various numerical methods were investigated by Arefin et al. (2020) [8]. According to Denis 2020, [9] when developing models in fields as diverse as engineering, economics, physics, astronomy, biology, chemistry, medicine, ecology, sociology, banking, and many more, differential equations rank high among the most significant mathematical tools.

This work presents an approach to addressing initial value problems using the Euler and Runge Kutta methods that does not rely on discretisation, transformation, or limiting assumptions. Most people's first encounter with numerical methods is with the Euler method. While intuitive and straightforward to express geometrically, the method lacks practicality and precision when applied to more complex functions.

Objective

To compare numerical solutions with the exact solutions between Euler method and Runge Kutta method.

Problem Formulation

An Initial Value Problem (IVP) in the context of ordinary differential equations (ODEs) consists of a differential equation coupled with an initial condition. The goal is to find a function $y(x)$ that satisfies both the differential equation and the initial condition at a specific point.

Consider the following general form of an IVP:

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$

Here:

- $\frac{dy}{dx}$ is the derivative of y with respect to x.
- $f(x, y)$ is a given function that describes
- the rate of change of y.
- $y(x_0) = y_0$ is the initial condition, specifying the value of y at $x = x_0$.

The solution to this IVP is the function $y(x)$ that satisfies the differential equation within a given interval, starting from the initial point (x_0, y_0).

To illustrate, consider the following nonlinear ODE:

$$\frac{dy}{dx} = -2xy^2 + 3x^2,\ y(0) = 1$$

In this example:

- The differential equation describes how y changes with respect to x.
- The initial condition $y(0) = 1$ specifies that the value of y is 1 when $x = 0$.

METHODOLOGY

Euler's

Euler's method is the simplest one-step method. It is basic explicit method for numerical integration of ordinary differential equations. Euler proposed his method for initial value problems (IVP) in 1768. It is first numerical method for solving IVP and serves to illustrate the concepts involved in the advanced methods. It is important to study because the error analysis is easier to understand. The general formula for Euler approximation is

$$y_{(n+1)}(x)=y_n(x)+hf(x_n,y_n),n=0,1,2,3,\cdots.$$

Runge Kutta

This method was devised by two German mathematicians, Runge about 1894 and extended by Kutta a few years later. The Runge Kutta method is most popular because it is quite accurate, stable and easy to program. This method is distinguished by their order in the sense that they agree with Taylor's series solution up to terms of h^r where r is the order of the method. It do not demand prior computational of higher derivatives of y(x) as in Taylor's series method. The fourth order Runge Kutta method (RK4) is widely used for solving initial value problems (IVP) for ordinary differential equation (ODE). The general formula for Runge Kutta approximation is.

$$y_{n+1}(x) = y_n(x) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), n = 0,1,2,3,\cdots$$

Where,

$$k_1 = hf(x, y),$$
$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right),$$
$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right),$$
$$k_4 = hf(x + h, y + k_3)$$

Error Analysis

There are two types of errors in numerical solution of ordinary differential equations. Round-off errors and Truncation errors occur when ordinary differential equations are solved numerically. Rounding errors originate from the fact that computers can only represent numbers using a fixed and limited number of significant figures. Thus, such numbers or cannot be represented exactly in computer memory. The discrepancy introduced by this limitation is call Round-off error. Truncation errors in numerical analysis arise when approximations are used to estimate some quantity. The accuracy of the solution will depend on how small we make the step size, h. A numerical method is said to be convergent if $\lim_{h\to 0} \max_{1\le n\le N}|y(x_n) - y_n| = 0$. Where $y(x_n)$ denotes the approximate solution and y_n denotes the exact solution. In this paper we consider two initial value problems to verify accuracy of the proposed methods. The Approximated solution is evaluated by using Mathematica software for two proposed numerical methods at given step size. The maximum error is defined by:

$$e_r = \max_{1\le n\le \text{steps}}(|y(x_n) - y_n|)$$

RESULTS

To solve this IVP numerically, we use the Euler method and the fourth-order Runge-Kutta method (RK4). Here's how we calculate the data for Table 1.

Calculation (Euler Method)

The Euler method approximates the solution by moving from one point to the next using the slope at the current point. Given a step size h, the formula is:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

Step-by-Step Calculation:
Initial Values: Start with $x_0 = 0, y_0 = 1$,, and step size h=0.1.
First Step:

- Calculate $f(x_0, y_0) = f(0,1) = -2(0)(1)^2 + 3(0)^2 = 0$.
- Update: $y_1 = y_0 + h \cdot 0 = 1 + 0.1 \cdot 0 = 1.000$.

Second Step (at $x_1 = 0.1$):

- Calculate $f(x_1, y_1) = f(0.1,1) = -2(0.1)(1)^2 + 3(0.1)^2 = -0.2 + 0.03 = -0.17$.
- Update: $y_2 = y_1 + 0.1 \cdot (-0.17) = 1 - 0.017 = 0.983$.

Continue: Repeat the process for each subsequent step until $x = 2$.

Calculation (Runge-Kutta Method (RK4)

The RK4 method uses a more sophisticated approach by calculating intermediate slopes ($k1, k2, k3$, k4) to get a better approximation of the solution.

Step-by-Step Calculation:

Initial Values: Start with $x_0 = 0, y_0 = 1$, and step size $h = 0.1$.

First Step:

- Calculate intermediate slopes:

$$k_1 = f(0,1) = 0$$

$$k_2 = f(0.05,1) = f(0.05,1 + 0.05 \times 0) = f(0.05,1) = -0.095$$

$$k_3 = f(0.05,1 - 0.0475) = -0.095$$
$$k_4 = f(0.1,1 - 0.0095) = -0.17$$

- Update:

$$y_1 = 1 + \frac{0.1}{6}(0 + 2(-0.095) + 2(-0.095) + (-0.17)) = 1 - 0.0096 \approx 0.9904$$

Second Step: Use y_1 and repeat the process until $x = 2$.

Tab. 1. Numerical Approximations and Exact Solutions.

x	Exact Solution $y(x)$	Euler Method $y(x)$	Error (Euler)	RK4 Method $y(x)$	Error (RK4)
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.5	1.0976	1.0833	0.0143	1.0971	0.0005
1.0	1.2307	1.1704	0.0603	1.2302	0.0005
1.5	1.4926	1.3487	0.1439	1.4918	0.0008
2.0	1.9759	1.5706	0.4053	1.9737	0.0022

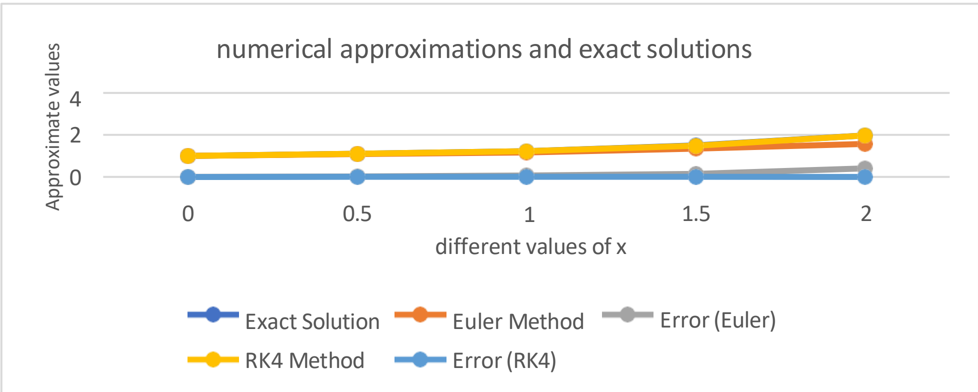

Fig. 1. Numerical Approximations and Exact Solutions

DISCUSSION ON RESULTS

The results presented in Table 1 highlight the performance of the Euler method and the fourth-order Runge-Kutta method (RK4) in solving the given initial value problem (IVP). The numerical approximations and corresponding errors were calculated for a nonlinear ordinary differential equation (ODE) using a step size of h=0.1, over the interval x∈ [0,2]. This section discusses the accuracy, computational implications, and practical considerations of using these two methods.

Accuracy of both methods

The Euler method is known for its simplicity and ease of implementation, which makes it a popular choice for solving ODEs. However, the results demonstrate a significant drawback: the accumulation of error as the calculation progresses. For instance, at x=0.5, the error is relatively small (0.0143), but by the time x reaches 2.0, the error has increased substantially to 0.4053. This growing error is characteristic of the Euler method, particularly when applied to nonlinear problems or when the step size is not sufficiently small.

In contrast, the RK4 method consistently produces results that are much closer to the exact solution, with significantly smaller errors at each step. The error for RK4 remains relatively stable across the interval, with only a slight increase from 0.0005 at x=0.5 to 0.0022 at x=2.0. This highlights one of the main advantages of the RK4 method: its ability to maintain high accuracy over larger intervals, making it suitable for problems where precision is critical.

The difference in accuracy between the Euler method and the RK4 method can be attributed to their underlying principles. The Euler method relies on a single tangent approximation, which can diverge from the actual solution path as x increases. On the other hand, the RK4 method computes the slope at multiple points within each step, effectively averaging these slopes to provide a more accurate approximation of the solution.

Computational Efficiency of both methods

However, the RK4 technique offers superior accuracy, but at the expense of heightened computing complexity. Each iteration of the RK4 technique necessitates the computation of four slopes (k_1, k_2, k_3, k_4), in contrast to the singular slope computation employed in the Euler method. The increased computational intensity of the RK4 technique may be a factor to consider in situations where there are constraints on computational resources or when the problem entails a substantial number of steps or equations.

Nevertheless, the computational expense associated with the RK4 technique is frequently rationalized due to its high level of precision, particularly in scenarios where little inaccuracies might result in substantial variances in the final answer. In instances of this nature, the supplementary computational demands imposed by the RK4 approach might obviate the necessity for even shorter step sizes, which would be imperative if employing the Euler method to attain comparable levels of precision. Accordingly, although the Euler approach may appear attractive due to its simplicity and efficiency, the RK4 technique frequently yields superior performance in terms of accuracy for a given computational workload

Practical Consideration

In practical application, the selection between the Euler technique and the RK4 method is contingent upon the particular demands of the given situation. The utilization of the Euler method may prove satisfactory in cases where there is minimal need for a high level of precision, or for initial analysis where a rapid approximation of the answer is necessary. The appeal of this approach lies in its simplicity of implementation and reduced computational expenses, rendering it particularly suitable for educational applications or scenarios with limited computer resources.

CONCLUSION

This study has compared the performance of the Euler method and the fourth-order Runge-Kutta method (RK4) in solving an initial value problem for a hypothetical nonlinear ordinary differential equation. The numerical results demonstrate that while the Euler method is simple and easy to implement, it suffers from significant error accumulation, particularly over larger intervals. In contrast, the RK4 method consistently provides highly accurate results, with much smaller errors, making it a more reliable choice for solving ODEs.

The choice between these methods should be guided by the specific needs of the problem. For problems where computational resources are limited or where a rough estimate of the solution is sufficient, the Euler method may be appropriate. However, for problems requiring high precision or involving nonlinear dynamics, the RK4 method is generally the better option due to its superior accuracy and stability.

In conclusion, while both methods have their merits, the RK4 method's ability to maintain accuracy with fewer steps makes it the preferred choice for solving complex or sensitive ODEs. Future research could explore the application of these methods to systems of differential equations, or the development of adaptive step-size techniques to further optimize the trade-off between accuracy and computational efficiency.

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